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# ON ULTRASONIC PROPAGATION THROUGH MERCURY IN TUBES

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## On Ultrasonic Propagation through Mercury in Tubes\*

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Some measurements have been made of the attenuation of ultrasonic waves in mercury as a function of tube diameter and the inner surface of the tube. A pulse technique was employed and attenuation was measured at 10.6 Mc/sec. by comparing successive echoes of the initial pulse. The results with smooth glass tubes are in qualitative (although not quantitative) agreement with the predictions of the Helmholtz theory for tubes large compared with the wavelength.

### INTRODUCTION

RECENTLY ultrasonic techniques<sup>1</sup> have been employed to supply faithful delay devices for electric signals when such delays may be as long as a few milliseconds. The electric signal is converted into an ultrasonic beam by some suitable transducer, such as a quartz crystal. The ultrasonic propagation takes place in a liquid-filled tube. For this reason some interest has been evoked in a more complete understanding of tubular propagation of ultrasonics.

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<sup>1</sup> Technical Series of the Massachusetts Institute of Technology Radiation Laboratory, Vol. 17, Chapter 7.

It has been shown<sup>2</sup> that some additional attenuation may be expected for an initially plane acoustic wave traveling down a tube with infinitely rigid walls. According to Crandall, there exists in the liquid a thin layer near the surface of the tube in which the amplitude of the sound falls rapidly to zero. The thickness of this layer is given by

$$d = (2\mu/\omega\rho)^{1/2},$$

where  $\mu$  is the coefficient of liquid viscosity,  $\omega$  is  $2\pi$  times the ultrasonic frequency, and  $\rho$  is the

<sup>2</sup> The problem of the damping exerted by a rigid-wall tube of large bore was first solved by Helmholtz. The analysis appears in Rayleigh *Theory of Sound* (Dover Publications, New York, 1945), Vol. II, p. 319. Several experiments on the velocity of sound in air in tubes have been performed to check the Helmholtz formula. For a discussion of the theory and references to experiments see I. B. Crandall's *Theory of Vibrating Systems and Sound* (D. Van Nostrand Company, Inc., New York, 1926), p. 229. The effect is also treated in P. M. Morse's *Vibration and Sound* (McGraw-Hill Book Company Inc., New York, 1936), p. 212.

density of the liquid. This layer exerts a drag on the motion in the main body of the tube, which would otherwise behave like a plane wave. By informal argument Crandall shows that the sonic amplitude undergoes tubular attenuation in addition to the free-space attenuation given by  $e^{-\alpha z}$  where  $\alpha = [1/av][(\omega\mu/2\rho)^{\frac{1}{2}}]$ . Here,  $a$  is the radius of the tube and  $v$  the velocity of free-space propagation. In the Appendix of this paper an analysis of this problem is presented and extended to take into account particle motion in the tube wall.

Besides loss due to viscous forces there is an additional heat conduction loss at the wall. This effect has been calculated by Kirchhoff<sup>2</sup> who has shown that, if thermal conductivity of the wall material greatly exceeds that of the fluid, the amplitude attenuation constant for tubular loss is

$$\alpha' = [1/av][(\omega/2\rho)^{\frac{1}{2}}][\mu^{\frac{1}{2}} + \{(\gamma-1)(K/c_v)\}^{\frac{1}{2}}],$$

where  $K$  is the thermal conductivity of the fluid,  $\gamma$  is the ratio of the specific heats, and  $c_v$  is the specific heat at constant volume.

Many ultrasonic delay devices employ mercury for the medium to fill the tubes. Since mercury will not adhere to surfaces with which it does not amalgamate, air will tend to be trapped between the mercury and wall in any surface irregularities which may exist. Such air pockets or layers are known to have a very profound effect on surfaces used to reflect ultrasonic beams<sup>3</sup> through 90°. In tubes the air at the surface of the mercury acts as a very low impedance. When this occurs, the situation is different from the case of the infinitely rigid wall and the theory of the viscous layer no longer applies. The propagation in the tube should depend markedly on the character of the tube surface, and it is difficult to predict the exact nature of the dependence.

For this reason it seemed worth-while to make an experimental study of tubular propagation and, in particular, quantitative measurements on attenuation in tubes of different internal diameters. The first measurements were made on

<sup>3</sup> We are indebted to Mr. H. J. McSkimin of the Bell Telephone Laboratories at Murray Hill for a chance to study the unpublished results of his research on reflectors and for interesting discussions on the subject.

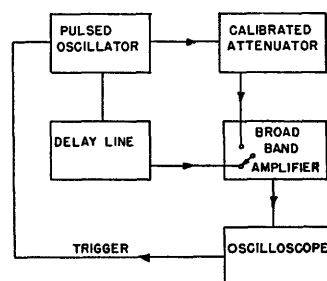


FIG. 1. Block diagram for voltage loss measurement.

steel tubes of varying degrees of roughness. Later, smooth glass tubes were measured in an effort to obtain some results under conditions which approximated more closely the assumptions of the Helmholtz theory.

### EXPERIMENTAL RESULTS

The measuring equipment (see Fig. 1) used consisted of an A-R Scope (DuMont 256B), a wide-band, 10-Mc/sec. amplifier with a passband of about 4-Mc/sec., a 70-ohm strip attenuator operated by toggle switches, and a pulsed 10-Mc/sec. oscillator, whose pulse duration could be varied from 0.4 to 2.5 microseconds. The internal trigger of the A-R Scope was used to initiate a video pulse which gated on the oscillator. The carrier pulse was fed simultaneously into the attenuator and the delay device. A switch at the input of the amplifier determined whether the output of the delay device or the attenuator was amplified and displayed on the A-R Scope. On the 4200- $\mu$  sec. sweep of this instrument many multiple echoes from the mercury column appeared simultaneously. By matching the height of the pulse through the attenuator with the height of any individual echo, one determined directly the voltage loss involved. The repetition rate of the A-R Scope was adjustable in the neighborhood of 300 cycles per second. The interval between pulses corresponded to time to traverse about 16 ft. of mercury, but there was no difficulty in measuring twice-around echoes, which increased the range of the measurements accordingly.

The choice for a measuring frequency in the neighborhood of 10 Mc/sec. was indicated as a compromise between lower frequencies where tubular attenuation would be insignificant and

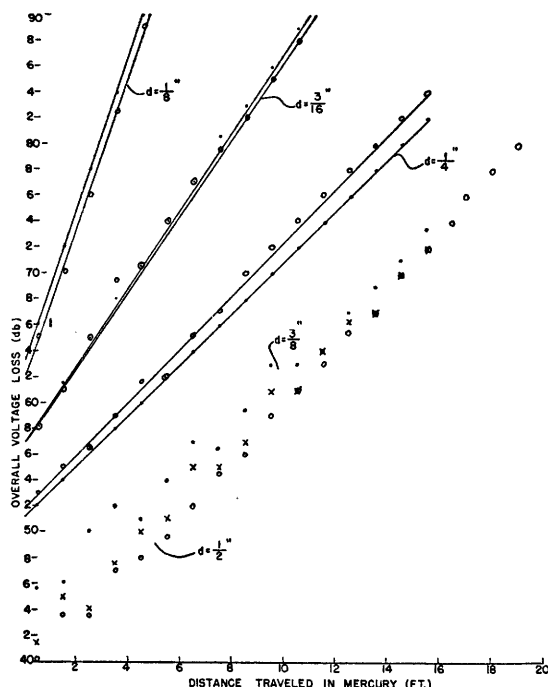


FIG. 2. Attenuation measurements with six-inch tubes. Solid dots, open circles, and crosses are used to distinguish between different runs.

higher frequencies where the number of echoes would be fewer and the effect might be masked by the uncertainties of the free-space attenuation (which varies as the square of the frequency). At 10 Mc/sec. the viscous layer theory predicts 1.00 db per ft. for a tube of  $\frac{1}{4}$ " bore, if one uses 1.40 centipoises for the viscosity of mercury at room temperature. Attenuation from thermal conductivity for the same tube might be as high as 2 db per ft., depending on wall material. Recent measurements indicate that the free-space attenuation is about  $1.25 \pm 0.15$  db per ft. at 10 Mc/sec. as against a theoretical value of 1.13 db per ft.

Crystals for 10-Mc/sec. operation are mechanically rugged and easily mounted without distortion. Four such crystals were used in the investigation, each mounted in its separate assembly, where it was held in place against a steel electrode by a gasket and clamp ring. The tubes under test were provided with parallel face plates at both ends against which the crystal assemblies could be bolted. Neoprene gaskets,  $\frac{1}{32}$ " thick, between assemblies and face plates, allowed final adjustment in operation to secure

accurate parallelism between the crystals. Within experimental error the crystal assemblies appeared to be completely interchangeable, and results with a given tube were reproducible with either pair of crystals.

Originally, it was decided to use five tubes, each 6" long, with inner diameters of  $\frac{1}{2}$ ",  $\frac{3}{8}$ ",  $\frac{1}{4}$ ",  $\frac{3}{16}$ ", and  $\frac{1}{8}$ ", respectively. To make comparison between these meaningful, the inner surface condition must be the same in every case. For the first experiment cold-rolled rod was used and carefully reamed to give a clean bore.

In measuring attenuation in the tubes with the pulse method the procedure was to record the intensity of successive echoes which arose from the multiple reflections at the two crystals. Since the crystals were backed by a dry metal electrode, practically none of the energy was coupled out. Also, the fraction of the energy used for the electrical signals was small enough to be negligible so that one could count on nearly lossless reflection from the back surface of each quartz crystal. The intensity of each echo was obtained by matching it in size with the undelayed signal through the calibrated attenuator. This method therefore gave the attenuation of each signal relative to the input pulse.

The results of these measurements are given in Fig. 2, where total attenuation for each of the five tubes is plotted against distance traveled in the tube. The intercepts of the straight lines through the data give the insertion loss for zero delay which may be considered as a voltage loss through mismatch at the crystals. These losses appear several db larger than one would calculate on the basis of the usual equivalent circuits for the crystal, considering the input impedance of the amplifier at the receiving crystal. (This impedance consisted of a 1200-ohm resistor and shunt coil to tune the crystal capacity at the frequency of operation, 10.6 Mc/sec.) One would expect the voltage at the receiver crystal to be proportional to the tube cross section and this prediction appears to be borne out within experimental accuracy, a fact which indicates a certain uniformity of excitation across the surface of the crystals.

For the smaller bore tubes,  $\frac{1}{4}$ ",  $\frac{3}{16}$ ", and  $\frac{1}{8}$ ", the signals fall off uniformly with distance, so that straight lines can be drawn through the

data. The slopes of these lines, from which attenuation per unit distance can be calculated, are quite reproducible to 0.15 db/ft. as indicated by the repeated runs. This reproducibility was not affected by changing the crystal assemblies but it was found necessary to clean the tube interiors between each run. For the larger bore tubes the data show a marked scatter, and it is not possible to draw straight lines through the data with certainty.

The next step in the investigation was to secure three additional tubes, 12" in length but otherwise similar in construction. These tubes had inner diameters of  $\frac{1}{2}$ ",  $\frac{3}{8}$ ", and  $\frac{1}{4}$ ", successively, and could be used alone or in series with the 6" tubes. The results with these units are given in Fig. 3. For the  $\frac{1}{4}$ " tubes the attenuation value shows no significant dependence on tube length. For the larger diameters the effects of lengthening the tube are to reduce the scatter and curvature of the points and give a lower value to the recorded attenuation. The straight lines are drawn to fit with the initial slope for the data taken with 18" tubing.

In Fig. 4, the five attenuation constants for the reamed tubes are plotted in db per ft. as solid dots against the reciprocal of the diameters. A reasonably smooth curve can be drawn through these points. It is, however, nearly parabolic in character instead of the straight line predicted by the Helmholtz formula.

This discrepancy proved to be all the more puzzling in the light of later experiments with these same tubes (i) after they had been lapped with 400 Carborundum and (ii) after they had been rough ground with 80 Carborundum. Under both of these last two conditions the attenuation proved considerably smaller for the  $\frac{3}{16}$ " and  $\frac{1}{8}$ " tubes than that for the originally reamed tubes. Unfortunately, time did not permit extensive investigation of this aspect of the research, so that only the qualitative result is stated here. It would seem that the "roughness" of the reamed surface should lie somewhere intermediate between that of the lapped and rough ground surfaces. Additional investigation is required to determine whether the larger attenuation observed with the reamed tubes can be attributed to the particular type of surface irregularity caused by the rotary motion of the reamer or to

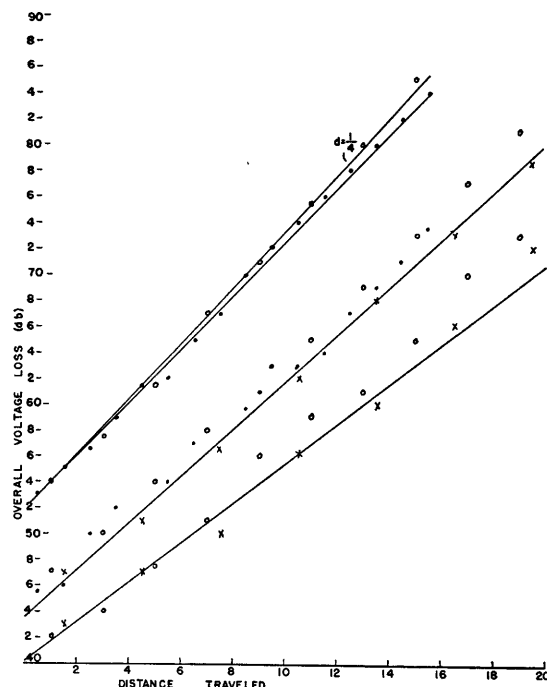


FIG. 3. Attenuation measurements with tubes of different lengths. Solid dots indicate measurements with 6" tubes, open circles with 12" tubes, and crosses with 18" tubes.

a condition of intermediate roughness at which the destructive interference in the mercury from the combined contacts with air and steel is at a maximum. In support of the latter hypothesis the work of investigators at the Bell Telephone Laboratories<sup>3</sup> with corner reflectors for ultrasonic beams indicate that such a mechanism can be responsible for considerable loss when the beam strikes the surface at a non-glancing angle.

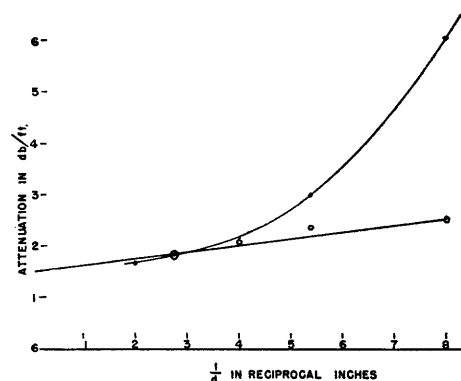


FIG. 4. Attenuation constants as a function of reciprocal tube diameter. Solid dots designate measurements in reamed tubes, open circles measurements with glass tubes.

Because the propagation with the steel tubes was complicated by the nature of the wall surface, it was decided to employ some smooth-wall glass tubes and try to maintain good acoustic contact between mercury and the tube. A series of measurements were next made with glass tubes of diameters  $\frac{3}{8}$ ",  $\frac{1}{4}$ ",  $\frac{3}{16}$ " and  $\frac{1}{8}$ ", and 6" in length. These were centered inside some of the steel tubes and the technique was otherwise identical. The attenuation constants in db per ft. are indicated by open circles in Fig. 4. (Considerable scatter again appeared in the run with  $\frac{3}{8}$ " diameter tube.) The data fit a straight line fairly well, giving a slope of 0.13 db per ft. per reciprocal inch as compared to the Helmholtz value of 0.25 db per ft. per reciprocal inch. Because of the low thermal conductivity of glass as compared to mercury, there will be no contribution to the attenuation from the thermal term in the Kirchhoff formula.

From the intercept of the straight line in Fig. 4 one can obtain a value for free-space attenuation in mercury, about 1.5 db per ft. This amounts to 1.3 db per ft. at 10 Mc/sec. which agrees within experimental error with the observed value quoted earlier in the paper. Another measurement of free-space attenuation was also made during the course of this investigation. The same crystal assemblies with active areas a  $\frac{1}{2}$ " diameter were bolted to a large tube with  $1\frac{3}{8}$ " inner diameter and 10" long. Measurement of successive echoes gave  $5.9 \pm 0.2$  db between signals, i.e., for 20" of mercury. From this the effect of diffraction must be subtracted—unfortunately, a large quantity. The diffraction loss between two parallel crystals of equal size has been calculated as a function of distance. For this instance it amounts to 1.8 db for one traversal of the tube which leaves 2.3 db for free-space attenuation in 20" of mercury. At 10 Mc/sec. one calculates an attenuation of 1.2 db per ft.

#### DISCUSSION

In this study of tubular attenuation for mercury it has been shown that the comparison of multiple echoes can be used to obtain a moderately accurate and reproducible measure of attenuation. With larger bore tubes it is necessary to increase their length correspondingly to reduce the scatter in the data and to obtain

attenuation values independent of tube length. Measurements have been made in reamed steel tubes, lapped steel tubes, and glass tubes.

Of these probably only the glass tubes were smooth enough to justify the assumption of acoustic contact at the tube wall, which is usually made in the theory. This attenuation was found to vary inversely as the tube diameter, in accordance with the Helmholtz formula, but the actual values were about half the theoretical prediction. In an appendix to this paper a calculation is given for deriving the Helmholtz formula and extending it to include the effect of particle motion in the tube wall. It turns out that this effect in most cases can be quite accurately taken into account by multiplying the expression for the tubular attenuation (infinitely rigid walls) by the factor  $[1 + (K/2\mu')]$  where  $K$  is the bulk modulus of the fluid and  $\mu'$  is the rigidity modulus of the tube wall. Though the rigidity modulus of the glass tubing was not known, it can be estimated that the correction factor in this case amounts to an increase of 30 to 50 percent. The direction of this correction is such as to increase the discrepancy between theory and experiment.

For steel tubes in good contact with the mercury the correction for particle motion in the wall amounts to about 17 percent, but the loss through the thermal conductivity of the steel must be included. The Kirchhoff formula (see Introduction) which takes this into account predicts an attenuation nearly three times that obtained from the simple Helmholtz formula which gives only the tubular attenuation from viscous forces. The disagreement here between theory and the experiments reported here is not too surprising, since it has been shown that the experimental results were very sensitive to the surface smoothness of the tubes employed. (On the other hand, the free-space value for the attenuation of mercury reported here and elsewhere appears to be in substantial agreement with the theory.)

The results of this investigation are somewhat of an exploratory nature. It is possible that further research will lead to a better correlation between theory and experiment. Specifically, the technique of vacuum filling the delay lines might yield some interesting results.

## APPENDIX

## Sonic Propagation in a Circular Tube

Though most of the results of this section on the problem of sonic propagation down a circular tube are already contained in the literature, it appears worth-while to present the analysis in this form. The treatment is extended to take into account the particle motion in the wall of the tube, and it turns out that the contribution to the sonic attenuation from this source can be taken quite readily into account.

In any medium which supports a shear there are two kinds of mechanical vibration which can be propagated, compressional and traverse. For the compressional disturbance the motion is irrotational and for the traverse waves the dilation vanishes. Since two media are involved in this problem, the fluid and the tube wall, there are four waves to be considered. The quantities which refer to the tube material are designated by primes,  $\rho'$ ,  $\lambda'$ , and  $\mu'$  are, respectively, the density and the usual isotropic elastic moduli of Lamé in this medium. In the liquid, density and bulk modulus are given by  $\rho$  and  $K$ . The isotropic coefficients of viscosity in the liquid are designated simply by  $\lambda$  and  $\mu$ . Cylindrical coordinates,  $r$ ,  $\theta$ , and  $z$  are used, where  $z$  is measured along the tube axis and  $r=a$  is inner surface of the tube wall.

It is useful at this point to introduce displacement potentials<sup>4</sup>  $\phi$  and  $\chi$  from which the displacements  $\xi_r$  and  $\xi_z$  can be obtained by differentiation as follows:

$$\begin{aligned}\xi_r &= \partial\phi/\partial r + \partial\chi/\partial z, \\ \xi_z &= \partial\phi/\partial z - \partial\chi/\partial r.\end{aligned}\quad (1)$$

Because all the quantities in this investigation are independent of  $\theta$ , it will not be necessary to introduce  $\xi_\theta$ . The potential  $\phi$  gives rise to a compressional wave, and  $\chi$  to a traverse (or shear) wave. In the wall the displacement potentials can be shown to satisfy the following equations:

$$\rho'(\partial^2\phi'/\partial t^2) = +(\lambda' + 2\mu')\nabla^2\phi', \quad (2A)$$

$$\rho'(\partial^2\chi'/\partial t^2) = +\mu'\nabla^2\chi'. \quad (2B)$$

<sup>4</sup>The seismologists have made extensive use of these potentials. For a summary of their treatment see Macelwane and Sohon, *Theoretical Seismology* (John Wiley and Sons, Inc., New York, 1936), p. 147.

In the fluid the corresponding equations are:

$$\rho(\partial^2\phi/\partial t^2) = +K\nabla^2\phi + (\lambda + 2\mu)\nabla^2(\partial\phi/\partial t), \quad (2C)$$

$$\rho(\partial^2\chi/\partial t^2) = +\mu\nabla^2(\partial\chi/\partial t). \quad (2D)$$

For investigation of behavior at a single frequency  $\omega/2\pi$ , the time-dependent factor can be written as  $e^{i\omega t}$  and will be omitted henceforth in the interest of conciseness. The spatial parts of the potentials will consist of sum of terms of the form  $\exp(-ik_z z)J_0(k_r r)$  and  $\exp(-ik_z z)N_0(k_r r)$  where  $J_0$  and  $N_0$  are the zero-order Bessel and Neumann functions, respectively. Problems depending on  $\theta$  would introduce higher order functions. In Table I are given sample terms for each of the four potentials appropriate to the physical demands of the problem under investigation. It is expected that  $\phi$  will be finite for  $r=0$  and that  $\phi'$  and  $\chi'$  fall off exponentially with  $r$ .

It follows from Eq. (2) that

$$\begin{aligned}k_z^2 + k_r^2 &= \omega^2/[K + i\omega(\lambda + 2\mu)] = k^2, \\ p_z^2 + p_r^2 &= \omega^2\rho/i\omega\mu = p^2, \\ (k_z')^2 + (k_r')^2 &= \omega^2\rho'/(\lambda' + 2\mu') = (k')^2, \\ (p_z')^2 + (p_r')^2 &= \omega^2\rho'/\mu' = (p')^2.\end{aligned}\quad (3)$$

In general, for a slightly viscous liquid the following inequalities hold between the above quantities:

$$|p| \gg |k| > p' > k'. \quad (4)$$

In Eqs. (5) below are given the displacement components as obtained from the potentials given in Table I. Use has been made of the relation  $[dJ_0(x)/dx] = -J_1(x)$ ,

$$\begin{aligned}\xi_r &= -k_r A \exp(-ik_z z)J_1(k_r r) \\ &\quad - ip_r B \exp(-ip_z z)J_0(p_r r), \\ \xi_z &= -ik_z A \exp(-ik_z z)J_0(k_r r) \\ &\quad + p_z B \exp(-ip_z z)J_1(p_r r),\end{aligned}\quad (5)$$

with similar expressions involving also  $N_0$  and  $N_1$  for the primed quantities.

The four boundary conditions to be satisfied at the inner surface of the tube ( $r=a$ ) are that the normal displacement, traverse displacement, normal pressure, and traverse shear all be continuous. It is apparent that these conditions can be satisfied by the displacements given in Eq. (5), and it is not necessary to introduce a sum of such expressions to satisfy the conditions at the walls. Moreover, the requirement that the said



TABLE I.

Potential	Sample Term
$\phi$	$A \exp(-ik_z z) J_0(k_r r)$
$\chi$	$B \exp(-ip_z z) J_0(p_z r)$
$\phi'$	$A' \exp(-ik_z' z) [J_0(k_r' r) + iN_0(k_r' r)]$
$\chi'$	$B' \exp(-ip_z' z) [J_0(p_z' r) + iN_0(p_z' r)]$

conditions are fulfilled for the length of tube insures that the factor containing the  $z$ -dependence be identical for all terms, i.e.

$$k_z = k_z' = p_z = p_z' = \gamma, \quad (6)$$

where  $\gamma$  is the propagation constant of the wave disturbance in the tube. Use is made of Eqs. (3) in simplifying the form boundary conditions written below:

Normal displacement:

$$-k_r A J_1(k_r a) - i\gamma B J_0(p_z a) \\ = -k_r' A' R_1(k_r' a) - i\gamma B' R_0(p_z' a). \quad (7a)$$

Tangential displacement:

$$-i\gamma A J_0(k_r a) + p_z B J_1(p_z a) \\ = -i\gamma A' R_0(k_r' a) + p_z' B' R_1(p_z' a). \quad (7b)$$

Tangential pressure:

$$\{(K + i\omega\lambda)k^2 + 2i\omega\mu k_r^2\} A J_0(k_r a) \\ + 2\gamma p_z \omega\mu B J_1(p_z a) \\ = \{\lambda'(k')^2 + 2\mu'(k_r')^2\} A' R_0(k_r' a) \\ - 2i\mu' \gamma p_z' R_1(p_z' a). \quad (7c)$$

Traverse shear:

$$i\omega\mu[2\gamma k_r A J_1(k_r a) - i(p_z^2 - \gamma^2) B J_0(p_z a)] \\ = \mu'[2\gamma' k_r' A' R_1(k_r' a) \\ - i\{(p_z')^2 - \gamma^2\} B' R_0(p_z' a)], \quad (7d)$$

where

$$R_m(z) = J_m(z) + iN_m(z).$$

$$\begin{vmatrix} -ik_r J_1(k_r a) & +\gamma & +k_r' & -\gamma \\ -\gamma J_0(k_r a) & +p_z & \gamma & p_z' \\ (\omega^2 \rho + 2i\omega\mu k_r^2) J_0(k_r a) & 2i\omega\mu \gamma p_z & -[\omega^2 \rho' + 2\mu'(k_r')^2] & +2\mu' \rho p_z' \\ 2i\omega\mu \gamma k_r J_1(k_r a) & \omega\mu p_z^2 & +2i\mu' k_r' \gamma & -2i\mu' \{\gamma^2 - (p_z')^2\} \end{vmatrix} = 0.$$

The following cases are of interest:

A. *Small acoustic impedance at the walls.*—Then the boundary conditions reduce to setting the normal pressure and traverse shear in the liquid equal to zero. From the lower left-hand corner of the original determinant one obtains

$$J_0(k_r a)/k_r J_1(k_r a) \\ = +4\gamma^2 \omega\mu/p_z [\omega^2 \rho + 2i\omega\mu k_r^2]. \quad (11)$$

It has been assumed that the tube dimensions are large compared to any of the wave-lengths involved, so that terms involving  $J_0/2a$  have been dropped.

If one rewrites Eq. (3) making use of inequalities (4)

$$k_r = (k^2 - \gamma^2)^{1/2}, \quad (8a)$$

$$p_z = (p^2 - \gamma^2)^{1/2} \simeq p = -[(1-j)/\sqrt{2}][(\omega\rho/\mu)^{1/2}], \quad (8b)$$

$$k_r' = \{(k')^2 - \gamma^2\}^{1/2} \simeq i\{\gamma - [(k')^2/2\gamma]\}, \quad (8c)$$

$$p_z' = \{(p')^2 - \gamma^2\}^{1/2} \simeq i\{\gamma - [(p')^2/2\gamma]\}. \quad (8d)$$

It is apparent that  $p_z$ ,  $k_r'$ , and  $p_z'$  are at least of order  $\gamma$ . Since  $(\gamma a)$  is large, as has just been pointed out, many of the Bessel and Neumann functions can be replaced by their asymptotic form<sup>5</sup> for large argument

$$J_m(z)_{z \rightarrow \infty} \rightarrow (2/\pi z)^{1/2} \cos[z - [(2m+1)/4]\pi], \\ N_m(z)_{z \rightarrow \infty} \rightarrow (2/\pi z)^{1/2} \sin[z - [(2m+1)/4]\pi]$$

and

$$R_m(z)_{z \rightarrow \infty} \rightarrow (2/\pi z)^{1/2} \exp[iz - [(2m+1)/4]\pi].$$

The signs of  $k_r'$  and  $p_z'$  have chosen to give exponentially damped functions in the wall, since all the incident energy from the liquid is outside the critical angle. The sign of  $p_z$  is chosen to give a positive imaginary component which makes the following simplification possible.

$$J_m(p_z a) \simeq [i^m/(2\pi p_z a)^{1/2}] \exp[-ip_z a + (i\pi/4)].$$

From Eqs. (7) a secular, determinantal equation can now be written to solve for  $k_r$ .

Since the quantity on the right is very small for slightly viscous fluids, the allowed values of  $k_r$  are nearly  $k_r = \alpha_i/a$  where  $\alpha_i$  are the roots of the zero-order, Bessel function. Since  $J_0(\alpha_i r/a)$  will show an oscillating character, a sum of such terms like the one that has been considered here

<sup>5</sup> P. M. Morse, *Vibration Sound* (McGraw-Hill Book Company, Inc., New York, 1936), p. 152.

will be needed to approximate a plane compressional wave traveling down the tube with uniform amplitude of displacement. This situation has already been treated in considerable detail by another investigator.<sup>6</sup> It has been shown that for this case one would not expect to find a tubular attenuation proportional to the length of path. It is conceivable that this situation is approximated by the rough tubes in which an air layer between mercury and most of the tube wall acts as a low acoustic impedance.

B. *An infinitely high acoustic impedance at the wall.*—This assumption is equivalent to setting the displacements in the liquid at the wall equal to zero. From the upper left-hand corner of the original determinant one obtains

$$k_r[J_1(k_r a)/J_0(k_r a)] = -i(\gamma^2/p_z). \quad (12)$$

Since the quantity on the right is very small, the allowed values of  $k_r$  are approximately  $\beta_i/a$  where  $\beta_i$  are the roots of the first-order Bessel Functions. Of particular interest is the case of  $\beta_0=0$ , when

$$k_r^2 \simeq -(\gamma^2/a)(2i/p_z) \\ \simeq -[(1+i)/a][(2\mu/\rho\omega)^{\frac{1}{2}}]\gamma^2. \quad (13)$$

The quantity  $k_r a$  is then so small that  $J_0(k_r r)$  is practically constant throughout the tube cross section. A one-term potential function in this case is sufficient to give a very good representation of a uniform plane wave moving down the tube. Only in the region of a thin film near the tube where the shear potential has an appreciable value does the displacement amplitude depart rapidly from its nearly constant value. This corresponds to the situation discussed by Crandall and mentioned in the introduction. The tubular attenuation formula can be obtained by substituting for  $k_r$  and  $k_z$  in the first equation in (2) and solving for

$$\gamma = [\omega^2 \rho / (K + i\omega(\lambda + 2\mu))]^{\frac{1}{2}} \\ \cdot [1 + ((i-1)\sqrt{2}/a)(\omega\mu/\omega^2 \rho)]^{-\frac{1}{2}} \\ = (\omega/v) + (1/ac)(\omega\mu/2\rho)^{\frac{1}{2}} \\ - i[(\omega^2/2\rho v^3)(\lambda + 2\mu) + (1/av)(\omega\mu/2\rho)^{\frac{1}{2}}]. \quad (14)$$

Here  $v$ , the free space velocity in the fluid, has been introduced for  $(K/\rho)^{\frac{1}{2}}$ . The imaginary part of  $\gamma$  gives the attenuation of the wave since the displacement varies as  $\exp(-i\gamma z)$ . The first

term in the square brackets gives the usual free space attenuation. (It is customary to assume that viscosity can play no role in pure dilation, i.e.,  $\lambda = -\frac{2}{3}\mu$ .) The second term in the brackets is the tubular attenuation.

C. *Approximate solution for the general solid tube.*—The secular equation (10) can also be solved without undue labor for those situations where it is permissible to set  $k_r' = p_z' = i\gamma$  (see Eq. (8)). For most of the harder wall materials  $p'/\gamma$  is less than one-half and for these cases the neglected part is less than one-eighth of the main term. With these approximations it becomes apparent that the upper right quarter of the secular determinant can be reduced to zeros by appropriate manipulation and the order of the determinant reduced from four to two.

$$\begin{vmatrix} -ik_r J_1(k_r a) & \gamma[1 + (K/2u')] \\ \gamma J_0(k_r a) & p_z \end{vmatrix} = 0. \quad (15)$$

Here viscous terms in  $\omega\mu$  have been neglected in comparison to the elastic moduli. For the smallest value of  $k_r$  one now has in place of Eq. (13)

$$k_r^2 = -(\gamma^2/a)(2i/p)[1 + (K/2u')]. \quad (16)$$

The extra factor  $[1 + (K/2u')]$  corresponds to an increase in attenuation of 17 percent for mercury in steel tube over the situation for an infinitely rigid tube. Apparently, the material of the tube exerts only slight effect on the viscous tubular attenuation as long as the bulk modulus in the fluid is considerably less than the rigidity modulus in the wall. In respect to the particular experiments discussed in the previous section the value of the rigidity modulus of the glass tubes was not known. A reasonable value for the correction term might range from 30 to 50 percent, which would increase the discrepancy between theory and experiment.

Of considerable interest is dependence of phase velocity in the tube on frequency. One would like to know what phase distortion in a transmitted signal pulse might be expected from such dispersion as may exist and what role the boundary conditions at the wall play in this problem. A convenient parameter to measure distortion is  $(\omega_0^2/\gamma)(d^2\gamma/d\omega^2)_{\omega=\omega_0}$ .

For Case A, where the acoustic impedance at the wall is negligible, one has

$$\gamma \simeq (\omega/v) + \frac{1}{2}(v/\omega)(\alpha_1/a)^2 \quad (17)$$

<sup>6</sup> Unpublished research by H. I. McSkimin. See reference 3.

for the phase of the mode associated with lowest root of  $J_0$ , namely  $\alpha_1$ . It follows that

$$(\omega_0^2/\gamma)(d^2\gamma/d\omega^2)_{\omega=\omega_0} = -(v\alpha_1/a\omega)^2.$$

For mercury in a tube of 1" bore at 10 Mc/sec. this quantity has the value  $0.32 \times 10^{-4}$ .

For Case B of high impedance wall material there is also some dispersion arising from the second term in Eq. (14).

$$\gamma = (\omega/v) + (1/av)(\omega\mu/2\rho)^{\frac{1}{2}}. \quad (18)$$

It follows for this case that

$$\begin{aligned} \omega_0^2/\gamma(d^2\gamma/d\omega^2)_{\omega=\omega_0} &= -(1/4\sqrt{2}a)(\mu/\omega\rho)^{\frac{1}{2}} \\ &= -0.63 \times 10^{-6}. \end{aligned}$$

For Case C this parameter of dispersion must be multiplied by  $[1 + (K/2u')]$ . Comparison of (17) and (18) indicate that one should expect much less difficulty with phase distortion in a ultrasonic delay device where the wall material did not appear as a low impedance acoustic element.

The amount by which the velocity of a signal (group velocity) in a smooth-walled tube (Cases B and C) is reduced from the free-space velocity is only one-half the amount by which the phase velocity is reduced. Since the group velocity is defined by  $v_g = (d\gamma/d\omega)^{-1}$ , it follows from Eqs. (14) and (18) that  $v_g = (v_T + v_B)/2$  to the first order in  $(1/a)(\mu/\omega\rho)^{\frac{1}{2}}$ .

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